PAST THE HIGHEST-WEIGHT, AND WHAT YOU CAN FIND THERE[*]

A.M. Semikhatov

Lebedev Physics Institute, Moscow 117924, Russia

The properties of highest-weight representations of the $N=2$ superconformal algebra in two dimensions can be considerably simplified when re-expressed in terms of relaxed $\widehat{s\ell}(2)$ representations. This applies to the appearance of submodules and hence, of singular vectors, and to the structure of the embedding diagrams and the BGG-type resolution. I also discuss the realization of these representations in the bosonic string, where the generalized DDK prescription amounts to the requirement that the representations have a charged singular vector, and the role of the fermionic screening operator.

INTRODUCTION

In this talk, I discuss how the crucial features of modules over the $N=2$ super-conformal algebra in two dimensions reformulate in a simpler way in terms of modules over the affine algebra $\widehat{s\ell}(2)$. The key statement, which can be advertised as

> the $N=2$ and affine-$s\ell(2)$ representations theories are "essentially equivalent"

was proved in [1]. Also analysed in that paper were the degenerations (reducibility patterns) of the $\widehat{s\ell}(2)$ representations (called the RELAXED Verma modules) that 'model' the $N=2$ Verma-like modules. In a parallel development, degenerations of the $N=2$ Verma modules were directly analysed in [2] (and, as a by-product, the problem of $N=2$ *sub*singular vectors was resolved).

In this talk, I will be much less formal than in those papers, and I will also discuss briefly some new developments [3,4] stemming from the results of [1,2]. These are aimed at constructing the BGG-type resolution [5] for the irreducible representations and, thus, at systematically deriving the characters and, on the other hand, at deriving the complete quantum symmetry of the $\widehat{s\ell}(2)$ fusion rules [6,7,8,9]. Before proceeding to more precise formulations, let us see whether the "essential equivalence" of the $N=2$ and $\widehat{s\ell}(2)$ representation theories comes as a news:

– on the one hand, the two algebras appear to have little in common, since one is a rank-2 (bosonic) affine Lie algebra, while the other is a rank-3 *super*algebra that is *not* an affine Lie algebra;

– on the other hand, the appearance of the two algebras in CFT is often 'correlated', they 'share' parafermionic theories, etc.

---

[*]Talk presented at the NATO Advanced Research Workshop on Theoretical Physics 'New Developments in Quantum Field Theory', June 14–20, 1997, Zakopane, and IV International Conference 'Conformal Field Theories and Integrable Models', Chernogolovka, June 23–27, 1997.

Surprisingly or not, establishing the equivalence requires introducing somewhat unusual $\widehat{s\ell}(2)$ representations; more precisely,

1. One considers an arbitrary complex level $k \in \mathbb{C} \setminus \{-2\}$ on the $\widehat{s\ell}(2)$ side.
2. On the $N = 2$ side, one considers the 'standard' representation category, which includes the Verma modules and their quotients, the (unitary and non-unitary) irreducible representations, etc., along with their images under the spectral flow (*twists*), for the central charge $\mathsf{c} \in \mathbb{C} \setminus \{3\}$.
3. *Modulo the spectral flow transform*, this $N = 2$ category is equivalent to the category of $\widehat{s\ell}(2)$ representations *of the type that has not been considered before* — the RELAXED highest-weight-type representations (and their twists).
4. On the other hand, the standard highest-weight-type $\widehat{s\ell}(2)$ representations (category $\mathcal{O}$ [10] *and their twists*) turn out to be related to a narrower category of (twisted) TOPOLOGICAL $N = 2$ Verma modules.

A statement regarding the equivalence of two categories can often be interpreted to the effect that there are two different languages to describe the same structure. In this way, any 'structural' result that is claimed about $N = 2$ Verma modules can in principle be seen in relaxed $\widehat{s\ell}(2)$ modules, and vice versa. As it may (and does) happen with equivalence of categories, however, an exciting point is that a number of fairly obvious facts about $\widehat{s\ell}(2)$ representations translate into the statements which are not quite obvious for the $N = 2$ representations. Thus, we see that the objects that are quite standard on the $N = 2$ side can be described in the $\widehat{s\ell}(2)$ terms by introducing a new type of modules, while only a subclass of $N = 2$ representations corresponds to the standard $\widehat{s\ell}(2)$ representations. Of a crucial importance is, therefore, the distinction between two different types of Verma-like modules over each algebra; for the $N = 2$ algebra, this distinction is masked due to an effect that we are going to discuss, while from the $\widehat{s\ell}(2)$ point of view, the distinction is much easier to see, and it can roughly be summarized by saying that in the relaxed Verma modules,

> *one goes past the highest-weight vector,*

which are going to describe in more detail. As a good illustration to the equivalence of categories, let me note that, while the $\widehat{s\ell}(2)$ representations where 'one goes past the highest-weight' may look somewhat unusual, THE SAME effects described in the $N = 2$ context are not considered unusual at all!

$\widehat{s\ell}(2)$ HIGHEST-WEIGHT REPRESENTATIONS

Let us begin with the $\widehat{s\ell}(2)$ algebra. We fix the level $k \neq -2$. Recall what one does when constructing a highest-weight-type module. The generators are broken into, roughly, two 'halves', one of which are declared annihilation operators with respect to a *highest-weight vector*, while the others *create* states, except for the 'Cartan' generator(s), whose eigenvalues simply 'label' the highest-weight vectors:

$$J_{\geq 0}^{+} |j, k\rangle_{s\ell(2)} = J_{\geq 1}^{0} |j, k\rangle_{s\ell(2)} = J_{\geq 1}^{-} |j, k\rangle_{s\ell(2)} = 0, \quad J_{0}^{0} |j, k\rangle_{s\ell(2)} = j |j, k\rangle_{s\ell(2)}, \quad (1)$$

where $j, k \in \mathbb{C}$. In the *Verma* module, by definition, there are no relations among the states produced by the creation operators from the highest-weight vector. That is,

the Verma module is *freely* generated from the highest-weight vector by the operators declared to be the creation ones. The structure of $\widehat{s\ell}(2)$ Verma modules is conveniently encoded in the *extremal diagram*

$$(2)$$

The states shown in the diagram are *extremal* in the sense that they have boundary values of the (charge, level) bigrading; all of the other states of the module should be thought of as lying in the interior of the angle in the diagram (on the rectangular lattice according to their (charge, level)). Finding a *submodule* in the Verma module can be (somewhat more schematically) represented as

$$(3)$$

Whenever one considers quotients of Verma modules, the extremal diagrams become 'smaller', as some of the states are eliminated from the module. All of such extremal diagrams, therefore, satisfy the following criterion:

$\boxed{\textit{Any straight line going through any state intersects the boundary on at least one end,}}$

which is formalized as follows: for any state $|X\rangle$ from the module,

$$\forall n \in \mathbb{Z}, \quad \exists N \in \mathbb{N}: \quad (\text{either} \quad (J_n^+)^N |X\rangle = 0 \quad \text{or} \quad (J_{-n}^-)^N |X\rangle = 0) \qquad (4)$$

However, this criterion also selects the so-called *twisted* modules. As regards twisted Verma modules, their extremal diagrams are 'rotations' of the above, e.g.:

$$(5)$$

In more formal terms, a *twisted Verma module* $\mathfrak{M}_{j,k;\theta}$ is freely generated by $J_{\leq\theta-1}^+$, $J_{\leq-\theta}^-$, and $J_{\leq-1}^0$ from a *twisted highest-weight vector* $|j,k;\theta\rangle_{s\ell(2)}$ defined by the conditions

$$J_{\geq\theta}^+ |j,k;\theta\rangle_{s\ell(2)} = J_{\geq1}^0 |j,k;\theta\rangle_{s\ell(2)} = J_{\geq-\theta+1}^- |j,k;\theta\rangle_{s\ell(2)} = 0\,,$$
$$\left(J_0^0 + \tfrac{k}{2}\theta\right) |j,k;\theta\rangle_{s\ell(2)} = j\,|j,k;\theta\rangle_{s\ell(2)}\,. \qquad (6)$$

The mapping that underlies the construction of twisted modules is known as the spectral flow transform[11]

$$\mathcal{U}_\theta: \quad J_n^+ \mapsto J_{n+\theta}^+\,, \qquad J_n^- \mapsto J_{n-\theta}^-\,, \qquad J_n^0 \mapsto J_n^0 + \tfrac{k}{2}\theta\delta_{n,0}\,, \qquad \theta \in \mathbb{Z}\,. \qquad (7)$$

Then,

$$\mathcal{U}_\theta\,|j,k\rangle_{s\ell(2)} = |j,k;\theta\rangle_{s\ell(2)}\,, \qquad |j,k\rangle_{s\ell(2)} = |j,k;0\rangle_{s\ell(2)}\,. \qquad (8)$$

A characteristic feature of extremal diagrams of $\widehat{s\ell}(2)$ Verma modules is the existence of the 'angle' that corresponds to the highest-weight vector. This will reappear in the topological $N\!=\!2$ Verma modules. We now consider the second main ingredient, the $N\!=\!2$ algebra and representations, after which we return to $\widehat{s\ell}(2)$ and "go past the highest-weight"; those representations won't have the angle in the extremal diagrams.

$N\!=\!2$ ALGEBRA AND REPRESENTATIONS

The $N=2$ superconformal algebra contains two fermionic currents, $\mathcal{Q}$ and $\mathcal{G}$, in addition to the Virasoro generators $\mathcal{L}$ and the $U(1)$ current $\mathcal{H}$. The commutation relations read as

$$
\begin{aligned}
[\mathcal{L}_m, \mathcal{L}_n] &= (m-n)\mathcal{L}_{m+n}\,, & [\mathcal{H}_m, \mathcal{H}_n] &= \tfrac{\mathsf{C}}{3}m\delta_{m+n,0}\,, \\
[\mathcal{L}_m, \mathcal{G}_n] &= (m-n)\mathcal{G}_{m+n}\,, & [\mathcal{H}_m, \mathcal{G}_n] &= \mathcal{G}_{m+n}\,, \\
[\mathcal{L}_m, \mathcal{Q}_n] &= -n\mathcal{Q}_{m+n}\,, & [\mathcal{H}_m, \mathcal{Q}_n] &= -\mathcal{Q}_{m+n}\,, & m,\ n \in \mathbb{Z}\,. \quad (9) \\
[\mathcal{L}_m, \mathcal{H}_n] &= -n\mathcal{H}_{m+n} + \tfrac{\mathsf{C}}{6}(m^2+m)\delta_{m+n,0}\,, \\
\{\mathcal{G}_m, \mathcal{Q}_n\} &= 2\mathcal{L}_{m+n} - 2n\mathcal{H}_{m+n} + \tfrac{\mathsf{C}}{3}(m^2+m)\delta_{m+n,0}\,,
\end{aligned}
$$

The element C is central; in representations, we will not distinguish between C and its eigenvalue $\mathsf{c}\in\mathbb{C}$, which it will be convenient to parametrize as $\mathsf{c} = 3\frac{t-2}{t}$ with $t\in\mathbb{C}\setminus\{0\}$. The $N\!=\!2$ spectral flow transform [12] acts as follows:

$$
\mathcal{U}_\theta : \begin{array}{llll}
\mathcal{L}_n & \mapsto & \mathcal{L}_n + \theta\mathcal{H}_n + \tfrac{\mathsf{c}}{6}(\theta^2+\theta)\delta_{n,0}\,, & \mathcal{H}_n & \mapsto & \mathcal{H}_n + \tfrac{\mathsf{c}}{3}\theta\delta_{n,0}\,, \\
\mathcal{Q}_n & \mapsto & \mathcal{Q}_{n-\theta}\,, & \mathcal{G}_n & \mapsto & \mathcal{G}_{n+\theta}\,,
\end{array}
\quad (10)
$$

Massive and topological $N\!=\!2$ modules

There exist two types of Verma-like modules over the $N=2$ algebra, which we call the massive and the topological ones; in the literature, the former are commonly considered as 'the' $N=2$ Verma modules, while the latter are precisely those $N=2$ modules that are in a good correspondence with the $\widehat{s\ell}(2)$ Verma modules from the previous section.

A *massive Verma module* $\mathcal{U}_{h,\ell,t}$ is freely generated by the generators $\mathcal{L}_{-m}$, $\mathcal{H}_{-m}$, $\mathcal{G}_{-m}$, $m \in \mathbb{N}$, and $\mathcal{Q}_{-m}$, $m \in \mathbb{N}_0$ (with $\mathbb{N}=1,2,\dots$ and $\mathbb{N}_0=0,1,2,\dots$) from a *massive highest-weight vector* $|h,\ell,t\rangle$ satisfying the following set of highest-weight conditions:

$$
\begin{aligned}
\mathcal{Q}_{\geq 1}\,|h,\ell,t\rangle &= \mathcal{G}_{\geq 0}\,|h,\ell,t\rangle = \mathcal{L}_{\geq 1}\,|h,\ell,t\rangle = \mathcal{H}_{\geq 1}\,|h,\ell,t\rangle = 0\,, \\
\mathcal{H}_0\,|h,\ell,t\rangle &= h\,|h,\ell,t\rangle\,, \qquad \mathcal{L}_0\,|h,\ell,t\rangle = \ell\,|h,\ell,t\rangle\,.
\end{aligned}
\quad (11)
$$

In the bigrading implied by (charge, level), or more precisely, by the eigenvalues of $(-\mathcal{H}_0, \mathcal{L}_0))$, the extremal diagram of a massive Verma module has the shape of a parabola for the simple reason that, having acted on the highest-weight vector with, say, $\mathcal{Q}_0$, applying the same operator once again gives identical zero, and 'the best one

can do' to construct a state with the extremal (charge, level) bigrading is to act with the $\mathcal{Q}_{-1}$ mode, etc.:

$$\tag{12}$$

An important fact is that all of the states on the extremal diagram satisfy the annihilation conditions

$$\mathcal{Q}_{-\theta+m+1} \approx \mathcal{G}_{\theta+m} \approx \mathcal{L}_{m+1} \approx \mathcal{H}_{m+1} \approx 0\,, \quad m \in \mathbb{N}_0 \tag{13}$$

for θ ranging over the integers, from $-\infty$ in the left end to $+\infty$ in the right end of the parabola.

Now, there can be *two* different types of Verma submodules in $\mathcal{U}_{h,\ell,t}$. In the language of extremal diagrams, these look like (with the discrete parabolas replaced by smooth ones for simplicity)

or

$$\tag{14}$$

In the first case, we have a *massive* Verma *sub*module, all of the states on its extremal diagram (as well as those on the extremal diagram of the module itself) satisfying the annihilation conditions from (11) for $\theta \in \mathbb{Z}$. In the other case, on the contrary, there is a distinguished state, marked with a •, that satisfies the annihilation conditions

$$\mathcal{Q}_{-\theta+m} \approx \mathcal{G}_{\theta+m} \approx \mathcal{L}_{m+1} \approx \mathcal{H}_{m+1} \approx 0\,, \quad m \in \mathbb{N}_0 \tag{15}$$

for a fixed $\theta \in \mathbb{Z}$. Such states will be referred to as (twisted) *topological highest-weight vectors*, and in the above context, as *topological singular vectors* (the $\theta = 0$ case being the 'untwisted' one). The precise definition is as follows.

A *twisted topological Verma module* $\mathfrak{V}_{h,t;\theta}$ is freely generated by $\mathcal{L}_{\leq -1}$, $\mathcal{H}_{\leq -1}$, $\mathcal{G}_{\leq \theta-1}$, and $\mathcal{Q}_{\leq -\theta-1}$ from a twisted *topological highest-weight vector* subjected to annihilation conditions (15), where, in addition,

$$\begin{aligned}
(\mathcal{H}_0 + \tfrac{\mathsf{c}}{3}\theta)\,|h,t;\theta\rangle_{\text{top}} &= h\,|h,t;\theta\rangle_{\text{top}}\,, \\
(\mathcal{L}_0 + \theta\mathcal{H}_0 + \tfrac{\mathsf{c}}{6}(\theta^2+\theta))\,|h,t;\theta\rangle_{\text{top}} &= 0\,.
\end{aligned} \tag{16}$$

A characteristic feature of the extremal diagram of a topological Verma module is the existence of a 'cusp', i.e. a state that satisfies stronger (the twisted topological)

5

highest-weight conditions than the other states in the diagram. As a result, the extremal diagram is narrower than that of a massive Verma module. In the $\theta = 0$ case for simplicity, the extremal diagram of $\mathcal{V}_{h,t} \equiv \mathfrak{V}_{h,t;0}$ reads as (with $|h,t\rangle_{\mathrm{top}} \equiv |h,t;0\rangle_{\mathrm{top}}$)

$$
\begin{array}{c}
|h,t\rangle_{\mathrm{top}} \\
\mathcal{G}_{-1} \swarrow \qquad \searrow \mathcal{Q}_{-1} \\
\bullet \qquad \bullet \\
\mathcal{G}_{-2} \swarrow \qquad \searrow \mathcal{Q}_{-2} \\
\bullet \qquad \bullet \\
\vdots \qquad \vdots
\end{array}
\tag{17}
$$

When taking quotients, the extremal diagrams may only become smaller, which allows us to formulate a criterion that automatically singles out the *topological highest-weight-type* modules (the corresponding $\mathcal{O}$-category). For any $n \in \mathbb{Z}$, by the 'massive' parabola $\mathcal{P}(n, X)$ running through a state $|X\rangle$, we understand the set of states

$$
\mathcal{Q}_{n-N} \ldots \mathcal{Q}_{n-1} \mathcal{Q}_n |X\rangle = 0 , \quad \mathcal{G}_{-n-M} \ldots \mathcal{G}_{-n-2} \mathcal{G}_{-n-1} |X\rangle = 0 , \qquad N, M \in \mathbb{N} . \tag{18}
$$

Then, a module belongs to the twisted 'topological' $\mathcal{O}$-category if, for any state $|X\rangle$,

$\boxed{\textit{any massive parabola intersects the extremal diagram of the module on at least one end}}$

which, again, means simply that the states (18) become zero in at least one branch, either for $N \gg 1$ or for $M \gg 1$.

The massive $N = 2$ Verma modules do not satisfy this criterion. However, in the massive case as well, one can formulate a criterion that does not allow the modules to become too wide: for any element $|X\rangle$,

$$
\forall n \in \mathbb{Z}, \quad \text{either} \ \ldots \mathcal{Q}_{n-2} \mathcal{Q}_{n-1} \mathcal{Q}_n |X\rangle = 0 \quad \text{or} \ \ldots \mathcal{G}_{-n-2} \mathcal{G}_{-n-1} \mathcal{G}_{-n} |X\rangle = 0 . \tag{19}
$$

In the next Section, we address the problem of finding the $\widehat{s\ell}(2)$ counterpart of the above $N = 2$ modules. We first map the generators and then investigate the representations.

FROM $N = 2$ TO $\widehat{s\ell}(2)$

An operator construction

We now use an operator construction allowing us to build the $\widehat{s\ell}(2)$ currents out of the $N = 2$ generators and a free scalar with the operator product $\phi(z)\phi(w) = -\ln(z-w)$. As a necessary preparation, we 'pack' the modes of the $N = 2$ generators into the corresponding fields, $\mathcal{T}(z) = \sum_{n \in \mathbb{Z}} \mathcal{L}_n z^{-n-2}$, $\mathcal{G}(z) = \sum_{n \in \mathbb{Z}} \mathcal{G}_n z^{-n-2}$, $\mathcal{Q}(z) = \sum_{n \in \mathbb{Z}} \mathcal{Q}_n z^{-n-1}$, and $\mathcal{H}(z) = \sum_{n \in \mathbb{Z}} \mathcal{H}_n z^{-n-1}$, and similarly with the $\widehat{s\ell}(2)$ currents. We also define vertex operators $\psi = e^\phi$ and $\psi^* = e^{-\phi}$. Then, for $\mathsf{c} \neq 3$,

$$
J^+ = \mathcal{Q}\psi , \qquad J^- = \tfrac{3}{3-\mathsf{c}} \mathcal{G}\psi^* , \qquad J^0 = -\tfrac{3}{3-\mathsf{c}} \mathcal{H} + \tfrac{\mathsf{c}}{3-\mathsf{c}} \partial\phi \tag{20}
$$

are the $\widehat{s\ell}(2)$ generators of level $k = \tfrac{2\mathsf{c}}{3-\mathsf{c}}$.[†]

[†]At the Conference, M. Halpern told me that such a mapping had been known to M. Peskin *et al.*, but I could not find the reference.

6

We also have a free scalar with signature -1, whose modes commute with the $\widehat{s\ell}(2)$ generators (20):

$$I^- = \sqrt{(k+2)/2}\,(\mathcal{H} - \partial\phi)\,. \tag{21}$$

The modes I_n^- generate a Heisenberg algebra. Then the module $\mathcal{F}_q^-$ is defined as a Verma module over this Heisenberg algebra with the highest-weight vector defined by $I_n^-|q\rangle^- = 0$, $n \geq 1$, and $I_0^-|q\rangle^- = q|q\rangle^-$.

Relating the representations

The behaviour of *representations* under operator constructions of this sort can be quite complicated.[‡] In our case, we take a topological Verma module $\mathcal{V}_{h,t}$ and *tensor* it with the module Ξ of the free scalar. The latter module is defined as $\Xi = \oplus_{n\in\mathbb{Z}}\mathcal{F}_n$, where $\mathcal{F}_n$ is a Verma module with the highest-weight vector $|n\rangle_\phi$ such that

$$\phi_m|n\rangle_\phi = 0\,,\ m \geq 1\,,\quad \psi_m|n\rangle_\phi = 0\,,\ m \geq n+1\,,\quad \psi_m^*|n\rangle_\phi = 0\,,\ m \geq -n+2\,, \tag{22}$$

and $\phi_0|n\rangle_\phi = -n|n\rangle_\phi$. We then have the following Theorem:
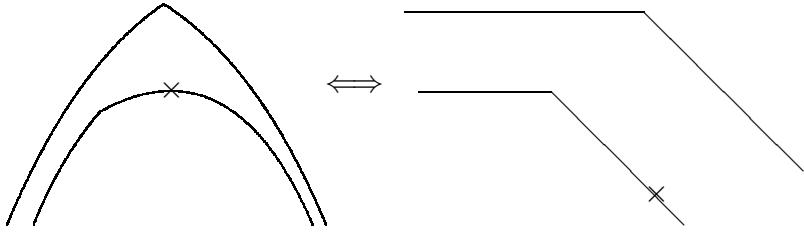
Theorem 1 ([1])

1. *There is an isomorphism of $\widehat{s\ell}(2)$ representations*

$$\mathcal{V}_{h,t} \otimes \Xi \approx \bigoplus_{\theta\in\mathbb{Z}} \mathfrak{M}_{-\frac{t}{2}h,t-2;\theta} \otimes \mathcal{F}^-_{\sqrt{\frac{t}{2}}(h+\theta)} \tag{23}$$

   where on the LHS the $\widehat{s\ell}(2)$ algebra acts by generators (20), while on the RHS it acts naturally on $\mathfrak{M}_{-\frac{t}{2}h,t-2;\theta}$ as on a twisted Verma module,

2. *A singular vector exists in $\mathcal{V}_{h,t}$ if and only if a singular vector exists in one (hence, in all) of the modules $\mathfrak{M}_{-\frac{t}{2}h,t-2;\theta}$, $\theta \in \mathbb{Z}$. Whenever this is the case, moreover, the submodules associated with the singular vectors, in their own turn, satisfy an equation of the same type as (23).*

The statement regarding singular vectors appeared, in a rudimentary form, in[17]. The theorem means that, as regards the existence and the structure of submodules, the topological $N = 2$ modules are *equivalent* to $\widehat{s\ell}(2)$ Verma modules: twisted topological Verma submodules appear simultaneously with $\widehat{s\ell}(2)$ Verma submodules[§], as

$$\tag{24}$$

[‡]Recall, for example, how the $\widehat{s\ell}(2)$ Verma modules are rearranged under the Wakimoto bosonization[13] — Wakimoto modules more or less 'interpolate' between Verma and contragredient Verma modules.
[§]in particular, the Ξ and $\mathcal{F}^-_{...}$ modules in (23) are truly 'auxiliary', since nothing can happen there that would violate the correspondence between submodules in topological $N = 2$ and $\widehat{s\ell}(2)$ Verma modules.

7

A common feature of $\widehat{s\ell}(2)$ and topological $N=2$ Verma modules is that all of them are generated from a state that satisfies stronger annihilation conditions than the other states in the extremal diagram[¶]. What *is* somewhat unusual about this correspondence, though, is the fact that on the $\widehat{s\ell}(2)$ side such a 'cusp' state[‖] satisfies the same annihilation conditions as the highest-weight state of the module, whereas on the $N=2$ side it satisfies *twisted* topological highest-weight conditions.

Thus, to the well-known $\widehat{s\ell}(2)$ singular vectors $|\text{MFF}(r,s,k)\rangle^{\pm}$, $r,s \in \mathbb{N}$, given by the construction of [14], there correspond the so-called *topological singular vectors*[18, 19] $|E(r,s,t)\rangle^{\pm}$, $t = k+2$, which satisfy twisted topological highest-weight conditions (15) with $\theta = \mp r$ respectively:

$$\mathcal{Q}_{\geq \pm r}|E(r,s,t)\rangle^{\pm} = \mathcal{G}_{\geq \mp r}|E(r,s,t)\rangle^{\pm} = \mathcal{L}_{\geq 1}|E(r,s,t)\rangle^{\pm} = \mathcal{H}_{\geq 1}|E(r,s,t)\rangle^{\pm} = 0. \quad (25)$$

As we see from the twist, the submodule generated from $|E^{\pm}(r,s,t)\rangle \in \mathcal{V}_{h,t}$ is the twisted topological Verma module $\mathfrak{V}_{h \pm r\frac{2}{t},t;\mp r}$. Equivalently, one may choose to describe the positions of $|E^{\pm}(r,s,t)\rangle \in \mathcal{V}_{h,t}$ in the (charge, level) lattice by using the *eigenvalues* of $\mathcal{H}_0$ and $\mathcal{L}_0$:

$$\mathcal{H}_0\left|E^{\pm}(r,s,t)\right\rangle = h_0^{\pm}\left|E^{\pm}(r,s,t)\right\rangle, \qquad \mathcal{L}_0\left|E^{\pm}(r,s,t)\right\rangle = \ell_0^{\pm}\left|E^{\pm}(r,s,t)\right\rangle, \quad (26)$$

then

$$h_0^{\pm} = h_0 \pm r, \qquad \ell_0^{\pm} = \ell_0 + \tfrac{1}{2}r(r+2s-1) \quad (27)$$

where h_0 and ℓ_0 are the *eigenvalues* of $\mathcal{H}_0$ and $\mathcal{L}_0$, respectively, on the highest-weight vector of the topological Verma module $\mathcal{V}_{h,t}$.

The topological singular vectors occur in the topological Verma module $\mathcal{V}_{h,t}$ whenever there exist $r,s \in \mathbb{N}$ such that the h parameter can be represented as $h = \mathsf{h}^{-}(r,s,t)$ or $h = \mathsf{h}^{+}(r,s,t)$, where

$$\mathsf{h}^{-}(r,s,t) = \tfrac{r+1}{t} - s, \qquad \mathsf{h}^{+}(r,s,t) = -\tfrac{r-1}{t} + s - 1 \quad (28)$$

The explicit construction for $N=2$ singular vectors can be found in[18, 19].

The idea regarding the correspondence between submodules can be developed in the direction of category theory. Very roughly, a category is a collection of *objects*, some of which may be related by *morphisms*. Taking the objects to be all the (twisted) topological $N=2$ Verma modules, the morphisms would have to be the usual $N=2$-homomorphisms. However, two *Verma* modules are related by a morphism only if one of the modules can be embedded into the other[**]. We have just seen that such

---

[¶]One *can* generate the same submodule from the state marked with a $\times$, but there are hardly any reasons to do so in the $\widehat{s\ell}(2)$ case. The point of [2] is that doing so in the $N=2$ case is equally inconvenient.

[‖]Assigning grade $-n$ to $\mathcal{Q}_n$ and grade n to $\mathcal{G}_n$, we see that every two adjacent arrows in diagram (17) represent the operators whose grades differ by 1, except at the cusp, where they differ by 2.

[**]Strictly speaking, this is so only for the "true" Verma modules, i.e., for the usual Verma modules on the $\widehat{s\ell}(2)$ side and the topological Verma modules on the $N=2$ side. These are precisely the modules we are interested in at the moment, however even for the massive $N=2$ modules and the corresponding relaxed $\widehat{s\ell}(2)$ modules that we introduce in what follows, the mappings with nontrivial kernels also agree in both categories.

embeddings — i.e., the occurrence of *sub*modules — are 'synchronized' between the topological $N = 2$ Verma modules and the $\widehat{s\ell}(2)$ Verma modules. In fact, there also exists a *functor* acting in the inverse direction, and one eventually concludes that the category $\mathcal{TOP}$ of topological $N = 2$ Verma modules is equivalent to the category $\mathcal{VER}$ of $\widehat{s\ell}(2)$ Verma modules. To be more precise, the appearance of the twist (the spectral flow transform) results in that this equivalence takes place only after one effectively factorizes over the spectral flows on either $N = 2$ and $\widehat{s\ell}(2)$ sides, see [1] for a rigorous statement. Anyway, an immediate consequence of this equivalence is that

> *Embedding diagrams of Verma modules are identical on the $N = 2$ and $\widehat{s\ell}(2)$ sides,*

where we are so far restricted to *topological* Verma modules on the $N = 2$ side. Since the $\widehat{s\ell}(2)$ embedding diagrams are well-known [15, 16], this spares us the job of deriving them in a less friendly environment of the $N = 2$ algebra.

As another consequence of the equivalence theorem, the results of [14, 15, 16] reformulate as follows:

> *a maximal submodule of a topological Verma module is either a twisted topological Verma module or a sum of two twisted topological Verma modules.*

Since every twisted topological Verma submodule is generated from a topological singular vector, this can be reformulated as the statement that *all singular vectors in topological Verma modules are the topological singular vectors* (the submodules being *freely* generated from these vectors). That the top-level representative of the extremal diagram of the $N = 2$ submodule in (24) satisfies only *massive*, rather that topological, highest-weight conditions does not, of course, change the fact that the submodule is *not* a massive one: the submodule is not *freely* generated from the top-level extremal state, as we saw in the criterion following (18). The attempts that have been made in the literature to find massive Verma modules inside the topological ones are erroneous.

Where do the massive $N = 2$ modules go?

Having seen that the topological $N = 2$ Verma modules are in a 'good' correspondence with $\widehat{s\ell}(2)$ Verma modules, we recall from (14) that this involves only a 'small' part of $N = 2$ Verma modules, whereas the massive $N = 2$ modules (the 'wide' ones) seem to have nowhere to go in the $\widehat{s\ell}(2)$ picture, since all of the capacities of the $\widehat{s\ell}(2)$ Verma modules are already used up to maintain the correspondence with the topological (the 'narrow') $N = 2$ Verma modules.

RELAXED $\widehat{s\ell}(2)$ VERMA MODULES

Solving the above problem requires introducing a new class of $\widehat{s\ell}(2)$ modules. These have a characteristic property that their extremal diagrams have *no 'cusps'* (no angles), which will be crucial for relating them to the massive $N = 2$ Verma modules (whose extremal diagrams have no cusps either). The recipe is to *relax* the annihilation conditions imposed on the highest-weight vector[††]:

$$J_0^+ \, |j, k\rangle_{s\ell(2)} = 0$$

[††]Yet the crossing out operation looked nicer in my transparencies.

For $\theta \in \mathbb{Z}$, the *twisted relaxed Verma module* $\mathfrak{R}_{j,\Lambda,k;\theta}$ is defined as follows. One takes the state $|j,\Lambda,k;\theta\rangle_{s\ell(2)}$ to satisfy annihilation conditions

$$J^+_{\geq\theta+1}\,|j,\Lambda,k;\theta\rangle_{s\ell(2)} = J^0_{\geq1}\,|j,\Lambda,k;\theta\rangle_{s\ell(2)} = J^-_{\geq-\theta+1}\,|j,\Lambda,k;\theta\rangle_{s\ell(2)} = 0\,. \qquad (29)$$

The module is generated from $|j,\Lambda,k;\theta\rangle_{s\ell(2)}$ by a free action of the operators $J^+_{\leq\theta-1}$, $J^-_{\leq-\theta-1}$, and $J^0_{\leq-1}$, and by the action of operators J^+_θ and $J^-_{-\theta}$ subject to the constraint

$$J^-_{-\theta}J^+_\theta\,|j,\Lambda,k;\theta\rangle_{s\ell(2)} = \Lambda\,|j,\Lambda,k;\theta\rangle_{s\ell(2)}\,. \qquad (30)$$

In addition, the j parameter is chosen such that

$$\left(J^0_0 + \tfrac{k}{2}\theta\right)|j,\Lambda,k;\theta\rangle_{s\ell(2)} = j\,|j,\Lambda,k;\theta\rangle_{s\ell(2)} \qquad (31)$$

(the normal ordering was chosen in (30) in order to facilitate the evaluation of the affine Sugawara dimension of the state).

Then, we can act on the highest-weight vector $|j,\Lambda,k;\theta\rangle_{s\ell(2)}$ with both J^+_0 and J^-_0, thereby producing new states

$$|j,\Lambda,k;\theta|n\rangle_{s\ell(2)} = \begin{cases} (J^-_{-\theta})^{-n}\,|j,\Lambda,k;\theta\rangle_{s\ell(2)}\,, & n < 0\,, \\ (J^+_\theta)^n\,|j,\Lambda,k;\theta\rangle_{s\ell(2)}\,, & n > 0\,, \end{cases} \qquad (32)$$

with $|j,\Lambda,k;\theta|0\rangle_{s\ell(2)} = |j,\Lambda,k;\theta\rangle_{s\ell(2)}$. As a result, the extremal diagram opens up to the straight angle; in the case of $\theta = 0$ it thus becomes

$$\cdots\ \bullet\xleftarrow{J^-_0}\bullet\xleftarrow{J^-_0}\bullet\xleftarrow{J^-_0}\star\xrightarrow{J^+_0}\bullet\xrightarrow{J^+_0}\bullet\xrightarrow{J^+_0}\bullet\ \cdots \qquad (33)$$

where all of the other states from the module correspond to points below the line. The $\star$ state is the above $|j,\Lambda,k;\theta=0\rangle_{s\ell(2)}$. We also define $|j,\Lambda,k|n\rangle_{s\ell(2)} = |j,\Lambda,k;0|n\rangle_{s\ell(2)}$, then the norms of these extremal states are given by

$$\left\||j,\Lambda,k|n\rangle_{s\ell(2)}\right\|^2 = \begin{cases} \prod_{i=0}^{-n-1}(\Lambda + 2(i+1)j - i(i+1))\,, & n \leq -1\,, \\ \prod_{i=0}^{n-1}(\Lambda - 2ij - i(i+1))\,, & n \geq 1\,. \end{cases} \qquad (34)$$

Thus, as we move either right or left along the extremal diagram, the norm becomes negative eventually. The negative-norm states can be factored away if it happens that the norm of one of the extremal states is exactly zero. This is the case whenever $\Lambda = \Lambda_{\mathrm{ch}}(p,j) \equiv p(p+1)+2pj$, $p \in \mathbb{Z}$; then the factors in (34) become $(1+i+p)(2j+p-i)$ and $(p-i)(1+i+2j+p)$ respectively. The corresponding zero-norm state

$$|C(p,j,k)\rangle_{s\ell(2)} = \begin{cases} (J^-_0)^{-p}\,|j,\Lambda_{\mathrm{ch}}(p,j),k\rangle_{s\ell(2)}\,, & p \leq -1\,, \\ (J^+_0)^{p+1}\,|j,\Lambda_{\mathrm{ch}}(p,j),k\rangle_{s\ell(2)}\,, & p \geq 0\,, \end{cases} \qquad (35)$$

then satisfies the Verma highest-weight conditions for $p \leq -1$ and the twisted Verma highest-weight conditions with the twist parameter $\theta = 1$ for $p \geq 1$. Thus, it is a *singular vector*, which can be quotiened away along with a tail of negative-norm states. For historical reasons [26, 1], states (35) are called *charged* singular vectors—they are an $\widehat{s\ell}(2)$ counterpart of the $N = 2$ singular vectors (shown schematically in the second diagram in (14)) that are called charged since [26].

Theorem 1 is now extended to

10

Theorem 2 (^{1})

1. *There is an isomorphism of $\widehat{s\ell}(2)$ representations*

$$\mathcal{U}_{h,\ell,t} \otimes \Xi \approx \bigoplus_{\theta \in \mathbb{Z}} \mathfrak{R}_{-\frac{t}{2}h,t\ell,t-2;\theta} \otimes \mathcal{F}^-_{\sqrt{\frac{t}{2}}(h+\theta)} \tag{36}$$

   where on the LHS the $\widehat{s\ell}(2)$ algebra acts by generators (20), while on the RHS it acts naturally on $\mathfrak{R}_{-\frac{t}{2}h,t\ell,t-2;\theta}$ as on a twisted relaxed Verma module.

2. *A singular vectors exists in $\mathcal{U}_{h,\ell,t}$ if and only if a singular vector exists in one (hence, in all) of the modules $\mathfrak{R}_{-\frac{t}{2}h,t\ell,t-2;\theta}$. Whenever this is the case, moreover, the respective submodules, in their own turn, satisfy an equation of the same type as (36) if these are massive/relaxed submodules, and Eq. (23) if these are topological/usual-Verma submodules.*

As follows from the notations, the parameters of the twisted relaxed Verma module on the RHS are $j = -\frac{t}{2}h$, $\Lambda = t\ell$, and $k = t - 2$. The simultaneous appearance of the massive/relaxed and topological/usual-Verma submodules can be illustrated as follows:
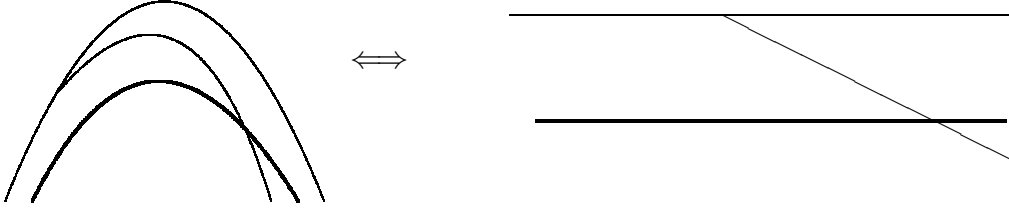
As a consequence,

> *embedding diagrams of massive $N = 2$ Verma modules are isomorphic to the embedding diagrams of relaxed $\widehat{s\ell}(2)$ Verma modules.*

The analysis of the latter is easier [3] because the affine-Lie algebra representation theory is available then and certain subdiagrams in the relaxed embedding diagrams are literally the standard $\widehat{s\ell}(2)$ embedding diagrams [15, 16]. Moreover, even though the relaxed $\widehat{s\ell}(2)$ Verma modules are not a 'classical' object in the representation theory of affine Lie algebras, the problem of enumerating submodules of relaxed Verma modules can be reduced, to a large extent, to analysing the standard $\widehat{s\ell}(2)$-embedding diagrams: for a given relaxed Verma module $\mathcal{R}$, one can find an *auxiliary* usual-Verma module $\mathcal{M}$ whose submodules are in a $1:1$ (or, in some degenerate cases, essentially in a $2:1$) correspondence with the relaxed Verma submodules in $\mathcal{R}$, see [3] for the classification and the detailed account [‡‡]. Recall that for Verma modules over the affine Lie algebras, the structure of the embedding diagrams is governed by the affine Weyl group; for the $N = 2$ algebra, we face the problem that it is not affine and, thus, no standard construction of an 'affine' Weyl group applies. Rather, it is the known [3] $N = 2$/relaxed-$\widehat{s\ell}(2)$ embedding diagrams that should suggest the appropriate representation of the Weyl group.

Let me also note that the $N = 2$/relaxed-$\widehat{s\ell}(2)$ *embedding* diagrams are made up of EMBEDDINGS, i.e., of mappings with trivial kernels. On the $N = 2$ side, this

---

[‡‡]Thus, the $N = 2$ extremal diagrams known in the literature[20, 21, 22] need being corrected already for the sole reason that they do not distinguish between topological and massive Verma modules. That the different types of $N = 2$ Verma-like modules were not recognized, complicates the analysis of degenerations of these modules in [23].

matter appears to have caused some confusion in the literature, because the existence of fermions was believed to lead to the vanishing of certain would-be embeddings. In the $\widehat{s\ell}(2)$ terms, however, this problem is obviously absent, hence it is but an artefact on the $N=2$ side as well. The vanishing of some compositions of the 'embeddings' observed previously is nothing but the manifestation of two facts: (i) the criterion that states (18) vanish for $N \gg 1$ or for $M \gg 1$ once $|X\rangle$ is inside a (twisted) topological Verma module, and (ii) the fact that every submodule generated from a charged singular vector is necessarily a twisted topological Verma module (similarly, and more transparent, on the $\widehat{s\ell}(2)$ side, where the charged singular vectors generate the usual (i.e., *not* relaxed) Verma modules, which obviously 'defermionizes' the whole picture).

Another construction which, in the affine case, reflects the structure of the Weyl group is the BGG resolution[5]. Translating the embedding diagrams into a BGG-type resolution requires more work in the $N=2$/relaxed-$\widehat{s\ell}(2)$ case because of the two types of submodules existing in the appropriate Verma-like modules. Unlike the embedding diagrams, the resolutions are constructed in terms of modules of only one type, therefore one would have to additionally resolve all the twisted topological Verma modules in terms of the massive Verma modules (or, in the $\widehat{s\ell}(2)$ terms, to resolve the usual-Verma modules in terms of a sequence of relaxed Verma modules with linearly growing twists). Constructing the resolution provides one with the tool for systematically deriving the $N=2$ characters and finding new representations for the known characters.

MASSIVE AND RELAXED MODULES IN THE BOSONIC STRING

According to the above equivalence Theorems, it is inessential in many respects whether one analyses $N=2$ Verma modules or relaxed $\widehat{s\ell}(2)$ Verma modules. In this section, we show how the above constructions (extremal states, massive Verma modules, etc.) arise naturally in the bosonic string.

In the noncritical bosonic string, one has the $N=2$ algebra realized as in[24, 25]. Applying the mapping described in the previous section, one recovers the corresponding realization of $\widehat{s\ell}(2)$ found in[17]. Let us describe in more detail the $N=2$ version of this construction. One starts with a matter theory represented by the energy-momentum tensor T with central charge $13-6/t-6t$ and tensors it with the bc ghosts and a (free) Liouville scalar. The resulting $N=2$ generators read as

$$
\begin{aligned}
\mathcal{T} &= T - t\partial\varphi\partial\varphi - (1+t)\partial^2\varphi - \partial bc - 2b\partial c\,, \quad \mathcal{H} = 2\partial\varphi + bc\,, \quad \mathcal{G} = b\,, \\
\mathcal{Q} &= -2b\partial cc - 2t\partial\varphi\partial\varphi c + 4\partial\varphi\partial c + 2Tc + (2-2t)\partial^2\varphi\,c + (1-\tfrac{2}{t})\partial^2 c\,,
\end{aligned}
\tag{37}
$$

where the Liouville OPE is chosen in a non-canonical normalization $\partial\varphi(z)\,\partial\varphi(w) = -(1/2t)\,1/(z-w)^2$. The representation space is then constructed as follows.

Constructing the representation

Each matter primary $|\Delta\rangle_{\mathrm{m}}$ of dimension Δ can be dressed into $N=2$ primaries either as

$$
\left| h, \Delta - \tfrac{1}{4}(2-2h-t+h^2 t), t; \theta \right\rangle_* = |\theta\rangle_{\mathrm{gh}} \otimes \left| e^{2t(-\frac{1}{2}-\frac{h}{2}-\frac{\theta}{t})\varphi} \right\rangle \otimes |\Delta\rangle_{\mathrm{m}} \tag{38}
$$

or by replacing $h \mapsto \frac{2}{t} - h$ in this formula (which does not change the $\mathcal{L}_0$ eigenvalue). Here,

$$|\theta\rangle_{\mathrm{gh}} = \begin{cases} b\,\partial b \ldots \partial^{-\theta-2}b\,|0\rangle_{\mathrm{GH}}\,, & \theta \le -2\,, \\ |0\rangle_{\mathrm{GH}}\,, & \theta = -1\,, \\ c\,\partial c \ldots \partial^{\theta}c\,|0\rangle_{\mathrm{GH}}\,, & \theta \ge 0\,, \end{cases} \tag{39}$$

are the ghost vacua in different pictures[27]. States (38) satisfy twisted massive highest-weight conditions (11). Further, in the tensor product of the matter Verma module with the ghosts (and the Liouville), each of the states (38) comes together with an infinite number of extremal states obtained by tensoring the same matter primary with ghost vacua in different pictures:

$$|\alpha\rangle_{\mathrm{gh}} \otimes \left|e^{2t(-\frac{1}{2}-\frac{h}{2}-\frac{\theta}{t})\varphi}\right\rangle \otimes |\Delta\rangle_{\mathrm{m}} \quad \text{and} \quad |\alpha\rangle_{\mathrm{gh}} \otimes \left|e^{2t(-\frac{1}{t}-\frac{1}{2}+\frac{h}{2}-\frac{\theta}{t})\varphi}\right\rangle \otimes |\Delta\rangle_{\mathrm{m}}\,, \quad \alpha \in \mathbb{Z}\,. \tag{40}$$

We thus see that in this realization,

> *choosing the ghost picture corresponds to traveling over the extremal diagram.*

The 'bosonization' (37) has the following effect: twisted *topological* highest-weight conditions (15) are satisfied whenever the dimension of a state (38) vanishes:

$$|h, 0, t; \theta\rangle_* = |h, t; \theta\rangle_{\mathrm{top}} \tag{41}$$

We will thus call $|h, \ell, t; \theta\rangle_*$ the *pseudomassive* (highest-weight) states. The *generalized DDK prescription* is that there be a twisted topological primary state among the extremal states (40). This is a condition on how the parameters in the tensor product of matter and Liouville are related: the matter dimension should be

$$\Delta(h, t) = \tfrac{1}{4}(2 - 2h - t + h^2 t)\,. \tag{42}$$

Then extremal states (40) become

$$|D(h, t, \theta, \alpha)\rangle \equiv |\alpha\rangle_{\mathrm{gh}} \otimes \left|e^{2t(-\frac{1}{2}-\frac{h}{2}-\frac{\theta}{t})\varphi}\right\rangle \otimes |\Delta(h,t)\rangle_{\mathrm{m}} = \left|h + \tfrac{2(\theta-\alpha)}{t}, \tfrac{(\theta-\alpha)(\alpha-\theta+1-ht)}{t}, t; \alpha\right\rangle_*\,.$$

As α runs over the integers, the $D(h, t, \theta, \alpha)$ states fill out the extremal diagram:

$$\tag{43}$$

The states in the upper curve are freely generated from $+\infty$, i.e., from any state where $\alpha \gg 1$. The state at $\alpha = \theta$ is the twisted topological highest-weight state $|h, t; \theta\rangle_{\mathrm{top}}$, with a twisted topological Verma submodule being generated from it. We also have another set of extremal states $D'(h, t, \theta, \alpha) = D(\frac{2}{t} - h, t, \theta, \alpha)$ constructed out of the same matter $|\Delta(h, t)\rangle_{\mathrm{m}}$.

A straightforward analysis shows that whenever $|D(h, t, \theta, \alpha_0)\rangle$ admits a singular vector for some $\alpha_0 \ne \theta$, each of the states $|D(h, t, \theta, \alpha)\rangle$, $\alpha \ne \theta$, admits a massive singular vector, while $|D(h, t, \theta, \theta)\rangle$ admits a topological singular vector. Then the states in the D- and D'-diagrams are given by

$$\begin{aligned} \left|D(\mathsf{h}^-(r, s, t), t, \theta, \alpha)\right\rangle &= |\alpha\rangle_{\mathrm{gh}} \otimes \left|e^{(-1-t-r+st-2\theta)\varphi}\right\rangle \otimes |\Delta_{r,s}(t)\rangle_{\mathrm{m}}\,, \\ \left|D(\mathsf{h}^+(r, s+1, t), t, \theta, \alpha)\right\rangle &= |\alpha\rangle_{\mathrm{gh}} \otimes \left|e^{(-1-t+r-st-2\theta)\varphi}\right\rangle \otimes |\Delta_{r,s}(t)\rangle_{\mathrm{m}}\,, \end{aligned} \qquad r,\,s \ge 1\,,$$

13

with $\mathsf{h}^\pm(r,s,t)$ defined in (28). It follows, moreover, that, using the above realization,

$\boxed{\text{\textit{topological } N\!=\!2 \textit{ singular vectors evaluate in terms of Virasoro representation states}}}$

as follows:

$$
\begin{array}{ccc}
E^+(r,1,t) & E^+(r,s+1,t) & E^-(r,s,t) \\
r \geq 1 & r,s \geq 1 & r,s \geq 1 \\
\downarrow & \searrow & \swarrow
\end{array} \qquad (44)
$$

Virasoro highest-weight state $\qquad\qquad$ Virasoro singular vector $(r,s|\Delta_{r,s})$

In view of the equivalence theorem, the same reduction holds for singular vectors in the usual-Verma $\widehat{s\ell}(2)$ modules, as known since long ago [29]. The fact that the $|E(r,1,t)\rangle^+$ topological singular vectors cannot be constructed out of the matter ones shows up in a different guise as the existence of a screening operator in the realization (37) (or, equivalently, a similar realization[17] of the $\widehat{s\ell}(2)$ algebra).

Mappings by the screening current

Whenever one uses an operator construction ('bosonization'), leading to some additional effects (e.g., vanishings) in representations (Eq. (41) in our case), one should expect the appearance of a screening current. We, indeed, have a *fermionic screening current* of the form[28, 4]

$$
F = b\,e^{t\varphi}\,\Psi_{12}\,, \qquad (45)
$$

where $\Psi_{1,2}$ is the '12' operator in the matter (Virasoro) sector. The $\Psi_{1,2}$ operator has two components that can be distinguished by picking out the following terms from the fusion relations:

$$
\Psi_{12}^\pm * |\Delta(h,t)\rangle_{\mathrm{m}} \sim |\Delta(h \pm 1, t)\rangle_{\mathrm{m}}\,. \qquad (46)
$$

Here, the highest-weight states may be understood as those in *Verma* modules over the Virasoro algebra. We now can construct the following action of the screening F on the pseudomassive modules over the $N\!=\!2$ algebra (we omit the integral which makes the screening *charge* out of the current):

$$
F^\pm : |D(h,t,\theta,\alpha)\rangle \mapsto \begin{cases} |D(h \pm 1, t, \theta, \alpha - 1)\rangle\,, & \alpha \geq \theta + 1\,, \\ 0\,, & \text{otherwise} \end{cases} \qquad (47)
$$

(which of the extremal states do, and which do not, vanish under the action of the screening, follows from a simple analysis of operator products). Then, for $\alpha \geq \theta + 1$,

$$
\begin{aligned}
F^- : & \quad \left|D(\mathsf{h}^-(r,s,t),t,\theta,\alpha)\right\rangle \;\mapsto\; \left|D(\mathsf{h}^-(r,s+1,t),t,\theta,\alpha-1)\right\rangle\,, \\
F^+ : & \quad \left|D(\mathsf{h}^+(r,s+1,t),t,\theta,\alpha)\right\rangle \;\mapsto\; \left|D(\mathsf{h}^+(r,s,t),t,\theta,\alpha-1)\right\rangle\,.
\end{aligned} \qquad (48)
$$

Next, we observe that the identities

$$
\left|D(\mathsf{h}^-(r,s,t),t,\theta_1,\alpha)\right\rangle = \left|D(\mathsf{h}^+(r,s+1,t),t,\theta_2,\alpha)\right\rangle\,, \quad \forall \alpha\,, \qquad (49)
$$

hold if and only if either $s = 0$, $r + \theta_1 = \theta_2$, or $t = \frac{r + \theta_1 - \theta_2}{s}$. For generic (non-rational) t, we can use the $s = 0$ case in order to connect the two series of mappings (48) together. Omitting the t parameter, we label the extremal diagrams spanned out by the $|D(\mathsf{h}^{\pm}(r, s, t), t, \theta, \alpha)\rangle$ states by the corresponding $\mathsf{h}^{\pm}(r, s)$ and the value of θ that gives the position of the topological highest-weight vector. We then have the following mappings of modules with the extremal diagrams (43)

$$\ldots \xrightarrow{F^+} \overbrace{}^{\mathsf{h}^+(r,2),r+\theta} \xrightarrow{F^+} \overbrace{}^{\mathsf{h}^+(r,1),r+\theta} \xrightarrow{F^-} \underbrace{}_{\mathsf{h}^-(r,0),\theta} \xrightarrow{F^-} \underbrace{}_{\mathsf{h}^-(r,1),\theta} \xrightarrow{F^-} \underbrace{}_{\mathsf{h}^-(r,2),\theta} \xrightarrow{F^-} \ldots \tag{50}$$

This sequence applies to those modules where the Virasoro part is taken to be *Verma* modules. We now investigate whether it is possible to go over from (50) to a similar sequence for *quotient* modules. According to (44), the massive $N = 2$ singular vectors would be factored away in all of the terms starting with and after $\mathsf{h}^-(r, 1)$ as soon as the Virasoro singular vectors are factored away. This allows one to define the F^- mappings between the *irreducible* representations. The same is true for the modules before and including $\mathsf{h}^+(r, 2)$. In the middle term $\mathsf{h}^-(r, 1)$, however, *there is no submodule* to factor over in the corresponding Virasoro Verma module. Yet, taking the composition $F^- \circ F^+$ allows us to conclude that

$$\boxed{\textit{for } t \in \mathbb{C} \setminus \mathbb{Q}, \textit{ there exists an exact sequence}}$$

$$\ldots \xrightarrow{F^+} \left|\overline{D(\mathsf{h}^+(r, 3, t), t, \theta + r)}\right\rangle \xrightarrow{F^+} \left|\overline{D(\mathsf{h}^+(r, 2, t), t, \theta + r)}\right\rangle \xrightarrow{F^- \circ F^+}$$
$$\left|\overline{D(\mathsf{h}^-(r, 1, t), t, \theta)}\right\rangle \xrightarrow{F^-} \left|\overline{D(\mathsf{h}^-(r, 2, t), t, \theta)}\right\rangle \xrightarrow{F^-} \ldots \tag{51}$$

where the bars indicate that the Virasoro singular vectors are declared to vanish (i.e., *irreducible* representations are taken in the matter sector in (38), (40), and similar formulae). The exact sequences of this sort (actually, those involving Verma modules) make up a part of the BGG resolution for the irreducible $N = 2$ representations in terms of the massive Verma modules. As I have mentioned, the BGG resolution allows one to find the characters; while for the affine Lie algebras the thus found character formulae reproduce the Weyl–Kač formula, the importance of the present approach consists in that, with the appropriate representation of the Weyl group not known, the BGG resolution has to be constructed directly from the corresponding embedding diagrams.

Another useful observation is that the above exact sequence is parallel to an exact sequence between representations of the quantum group $s\ell(2|1)_q$ (for q not a root of unity in accordance with the above choice of generic t), which is not a coincidence[4]. The $s\ell(2|1)_q$ embeddings are also performed by 'fermionic' singular vectors, with a due analogue of the $F^- \circ F^+$ composition in the center. In fact, generating the quantum $s\ell(2|1)_q$ symmetry involves other screenings in addition to F, however these do not have to be explicitly introduced in the present approach, where we do not bosonize the matter (energy-momentum tensor T) through *free* fields. The $s\ell(2|1)$ quantum group has long been suspected to govern the complete $\widehat{s\ell}(2)$ fusion rules[7], however the presently observed symmetry is only $osp(1|2)_q$. It may be expected that the model with a $s\ell(2|1)_q$-symmetric fusion may be constructed by developing the above observations.

REFERENCES

1. B.L. Feigin, A.M. Semikhatov, and I.Yu. Tipunin, Equivalence between chain categories of representations of affine $s\ell(2)$ and $N = 2$ superconformal algebras, `hep-th/9701043`.
2. A.M. Semikhatov and I.Yu. Tipunin, The structure of Verma modules over the $N = 2$ superconformal algebra, hep-th/9704111, to appear in *Commun. Math. Phys.*
3. A.M. Semikhatov and V.A. Sirota, Embedding diagrams of $N = 2$ and relaxed-$\widehat{s\ell}(2)$ Verma modules, hep-th/9712nnn.
4. B.L. Feigin, A.M. Semikhatov, and I.Yu. Tipunin, unpublished.
5. I. Bernshtein, I. Gelfand, and S. Gelfand, *Funk. An. Prilozh.* 10 (1976) 1.
6. H. Awata and Y. Yamada, Fusion rules for the fractional level sl(2) algebra, *Mod. Phys. Lett. A*7 (1992) 1185.
7. B. Feigin and F. Malikov, Modular functor and representation theory of $\widehat{\mathfrak{sl}}_2$, *Cont. Math. 202* "Operads: Proceedings of Reneissance Conferences".
8. O. Andreev, Operator algebra of the $SL(2)$ conformal field theories, *Phys. Lett. B*363 (1995) 166.
9. J.L. Petersen, J. Rasmussen, and M. Yu, Fusion, crossing and monodromy in conformal field theory based on $SL(2)$ current algebra with fractional level, *Nucl. Phys. B*481 (1996) 577-624.
10. V.G. Kač Infinite Dimensional Lie Algebras, *Cambridge University Press*, 1990.
11. K. Bardakçi and M.B. Halpern, *Phys. Rev. D*3 (1971) 2493.
12. A. Schwimmer and N. Seiberg, Comments on the $N = 2$, $N = 3$, $N = 4$ superconformal algebras in two-dimensions, *Phys. Lett. B*184 (1987) 191.
13. B.L. Feigin and E.V. Frenkel, Representations of affine Kac–Moody algebras and bosonization, in: *Physics and Mathematics of Strings*, eds. L. Brink, D. Friedan, and A.M. Polyakov.
14. F.G. Malikov, B.L. Feigin, and D.B. Fuchs, Singular Vectors in Verma Modules over Kač–Moody Algebras, *Funk. An. Prilozh.* 20 N2 (1986) 25.
15. A. Rocha-Caridi and N.R. Wallach, Highest weight modules over gaded lie algebras: rsolutions, filtrations, and character formulas, *Trans. Amer. Math. Soc.* 277 (1983) 133-162.
16. F. Malikov, Verma modules over rank-2 Kač–Moody algebras, *Algebra i Analiz* 2 No. 2 (1990) 65.
17. A.M. Semikhatov, The MFF singular vectors in topological conformal theories, *Mod. Phys. Lett. A*9 (1994) 1867.
18. A.M. Semikhatov and I.Yu. Tipunin, *Int. J. Mod. Phys. A*11 (1996) 4597.
19. A.M. Semikhatov and I.Yu. Tipunin, All Singular vectors of the $N = 2$ superconformal algebra via the algebraic continuation approach, hep-th/9604176.
20. E.B. Kiritsis, *Int. J. Mod. Phys. A* (1988) 1871.
21. V.K. Dobrev, Phys. Lett. B186 (1987) 43.
22. Y. Matsuo, *Prog. Theor. Phys.* 77 (1987) 793.
23. M. Dörrzapf, *Commun. Math. Phys.* 180 (1996) 195.
24. B. Gato-Rivera and A.M. Semikhatov, *Phys. Lett. B*293 (1992) 72.
25. M. Bershadsky, W. Lerche, D. Nemeschansky, and N.P. Warner, *Nucl. Phys. B*401 (1993) 304.
26. W. Boucher, D. Friedan, and A. Kent, *Phys. Lett. B*172 (1986) 316.
27. D.H. Friedan, E.J. Martinec, and S.H. Shenker, *Nucl. Phys. B*271 (1986) 93.
28. I. Tipunin, unpublished.
29. A.Ch. Ganchev and V.B. Petkova, *Phys. Lett. B*293 (1992) 56–66; *Phys. Lett. B*318 (1993) 77–84.